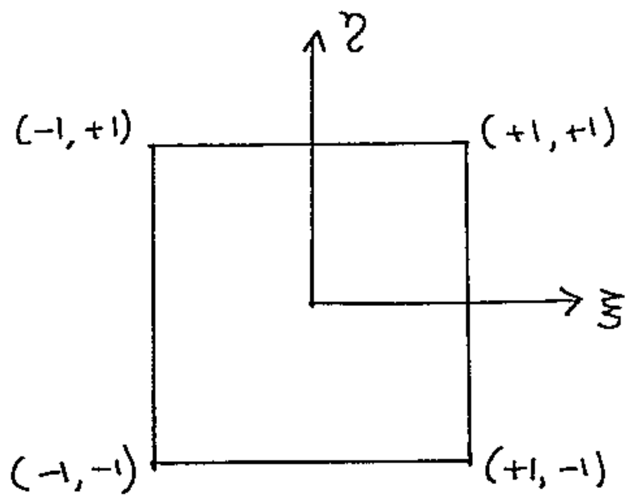
Lecture No. 14

Two Dimensional Basis Functions Quadrilaterals

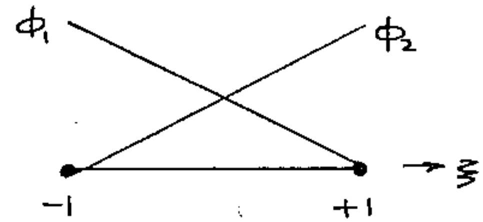
Let's assume that we can define a rectangular "unit" element as follows:



- Lagrangian basis Functions have C_o Functional Continuity.

  One way to generate 2-D basis functions is to take the product of two 1-D basis functions, one written for each coordinate direction. This approach can be applied for linear, quadratic and cubic Lagrange and for Hermite cubic.

- <u>Linear Lagrange</u> (Bi-Linear Lagrange Quadrilateral)
  Let's apply the described procedure to develop a linear Lagrange 2-D element.



For 1-D $\phi_1(\xi) = \frac{1}{2}(1-\xi)$

$$\phi_2(\xi) = \frac{1}{2}(1+\xi)$$

In the second direction we will have

$$\phi_1(\eta) = \frac{1}{2}(1-\eta)$$

$$\phi_2(\eta) = \frac{1}{2}(1+\eta)$$

Let's now take the products of these 4 functions to find the 2-D functions:

$$\phi_1(\xi,\eta) = \frac{1}{2}(1-\xi)\frac{1}{2}(1-\eta)$$

$$\phi_2(\xi,\eta) = \frac{1}{2}(1+\xi)\frac{1}{2}(1-\eta)$$

$$\phi_3(\xi,\eta) = \frac{1}{2}(1+\xi)\frac{1}{2}(1+\eta)$$

$$\phi_4(\xi,\eta) = \frac{1}{2}(1-\xi)\frac{1}{2}(1+\eta)$$

Therefore we have 4 functions. These correspond to 4 nodes which are located at the corners.

$\phi_1 = 1@(-1,-1)$ and zero at all other corners

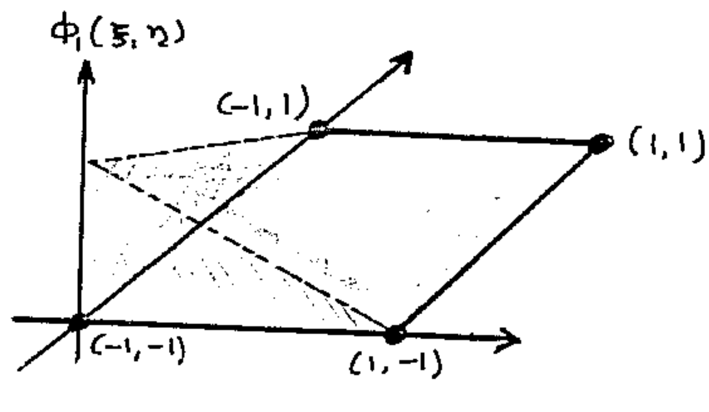
$\phi_2 = 1@(1,-1)$ and zero at all other corners

$\phi_3 = 1@(1,1)$ and zero at all other corners

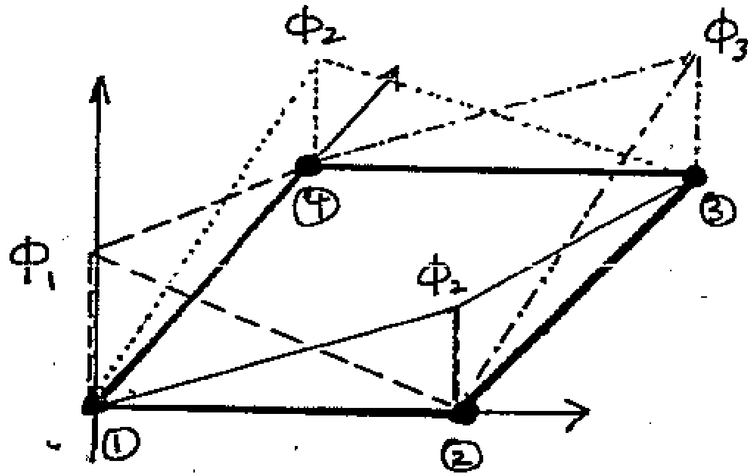
$\phi_4 = 1@(-1,1)$ and zero at all other corners

Consider:

$$\phi_1(\xi,\eta) = \frac{1}{4}(1 - \eta - \xi + \eta\xi)$$

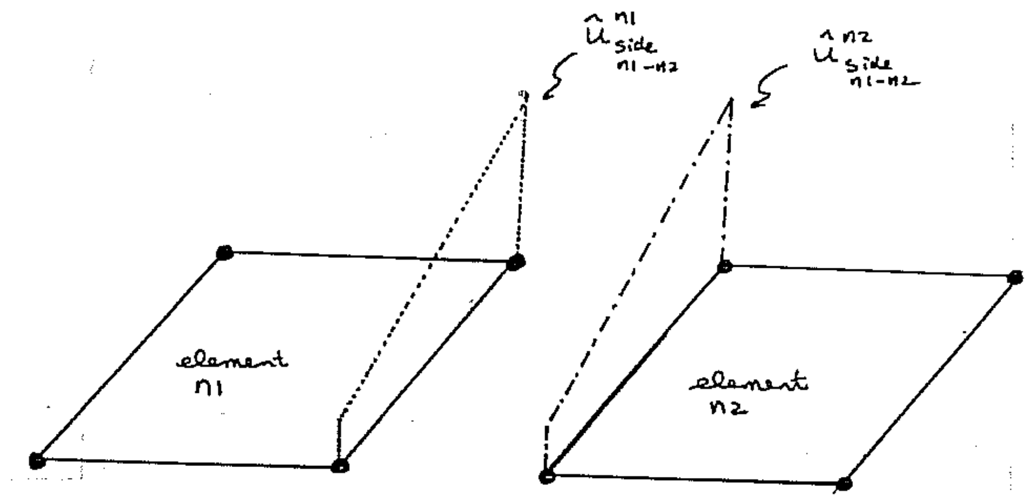


The variation along the element sides is linear. However within the element, the variation is not purely linear due to the cross term $\xi\eta$



- Inter-element continuity at any point along any side has been assured due to the fact that the functional variation in the adjacent element has been defined in the same way and with the same coefficients.

$$\hat{u} = \underline{\phi}\,\underline{u}^{(n)} = \phi_1 u_1^{(n)} + \phi_2 u_2^{(n)} + \phi_3 u_3^{(n)} + \phi_4 u_4^{(n)}$$



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Variation along side only:

$$\underset{n1-n2}{\hat{u}_{side}^{n1}} = \phi_2 u_2^{(n1)} + \phi_3 u_3^{(n1)}$$

Variation along side only:

$$\underset{n1-n2}{\hat{u}_{side}^{n2}} = \phi_1 u_1^{(n2)} + \phi_4 u_4^{(n2)}$$

However due to functional continuity constraint

$$u_2^{(n1)} = u_1^{(n2)}$$

$$u_3^{(n1)} = u_4^{(n2)}$$

Thus in general

$$\underset{n1-n2}{\hat{u}_{side}^{n1}} = \underset{n1-n2}{\hat{u}_{side}^{n2}}$$

Therefore the variation along the entire side is the same for <u>both</u> elements.

- Note that the general variation along all boundaries and within the element is given by $\hat{u}^{(n)} = \underline{\phi}\,\underline{u}^{(n)}$. The formulae given above are simply simplified down for the given side.
- Thus functional continuity is assured along all inter-element boundaries due to definition of the interpolating functions as well as the assurance that the nodal constants (i.e. $u_i^{(n)}$) are forced equal for shared nodes.

Quadratic Lagrange (Bi-quadratic Lagrange Quadrilateral)

For 1-D elements in ξ-direction

$$\phi_1(\xi) = \xi(\xi - 1)/2$$

$$\phi_2(\xi) = 1 - \xi^2$$

$$\phi_3(\xi) = \xi(1 + \xi)/2$$

Thus in the η-direction we define:

$$\phi_1(\eta) = \eta(\eta - 1)/2$$

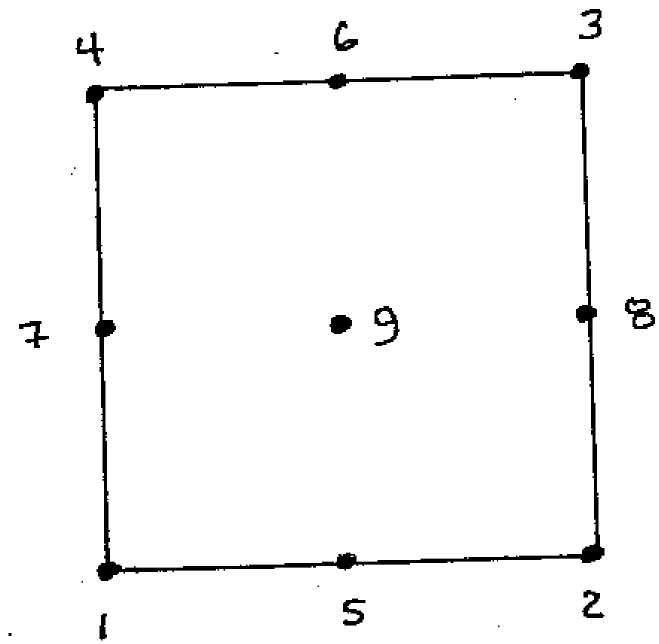
$$\phi_2(\eta) = 1 - \eta^2$$

$$\phi_3(\eta) = \eta(1 + \eta)/2$$

- Now we take the product of these functions.

  This yields 9 functions which are associated with 9 nodes.

  8 functions for side nodes. These equal unity at one side node and zero at all other nodes.



4 corner nodes:

$$\phi_i = \frac{1}{4}\xi\xi_i(1+\xi\xi_i)\eta\eta_i(1+\eta\eta_i) i = 1,4$$

4 mid-side nodes:

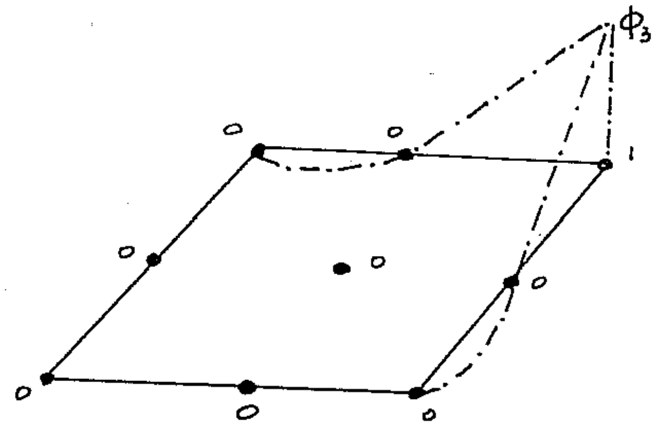
$$\xi_{i+4} = 0 \quad \phi_{i+4} = \frac{1}{2}\eta\eta_{i+4}(1+\eta\eta_{i+4})(1-\xi^2) \quad i = 1,2$$

$$\eta_{i+6} = 0 \quad \phi_{i+6} = \frac{1}{2}\eta\eta_{i+6}(1+\eta\eta_{i+6})(1-\eta^2) \quad i = 1,2$$

The 9th function is defined at the center point (0,0):

$$\phi_9 = (1-\xi^2)(1-\eta^2)$$

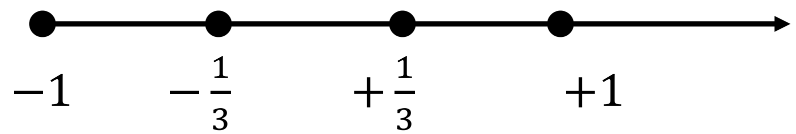
This function equals unity at the center point (0,0) and zero at all side nodes.



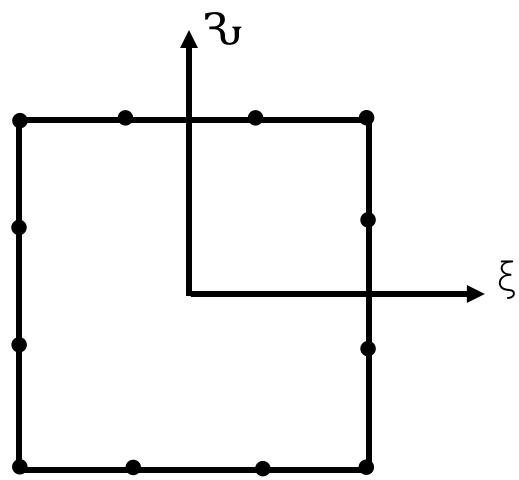
- Thus each interpolating function is defined to be zero at all nodes except for one, where it will equal unity.
- We now have a quadratic variation along the sides. Therefore full functional continuity between inter-element boundaries is assured.
- We have up to 4th order terms in the interior of the element.

Cubic Lagrange (Bi-cubic Lagrange Quadrilateral)

1-D element has 4 interpolating functions and 4 associated nodes



2-D element will have 16 interpolating functions and 16 nodes



Cubic variation along element sides.

6^{th} order polynomial within the element interior.

Formal Derivation of Interpolating Polynomials for Quadrilateral Elements

Steps:

1. Define the general form of the interpolating polynomial (e.g. bi-linear, bi-quadratic, etc.) and the location of the nodes. It is very useful to utilize Pascal's Triangle in this definition process.

Pascal's Triangle: Provides a simple pattern for characterizing complete polynomials for quadrilateral elements and associating the requisite nodal points with the element.

k

0

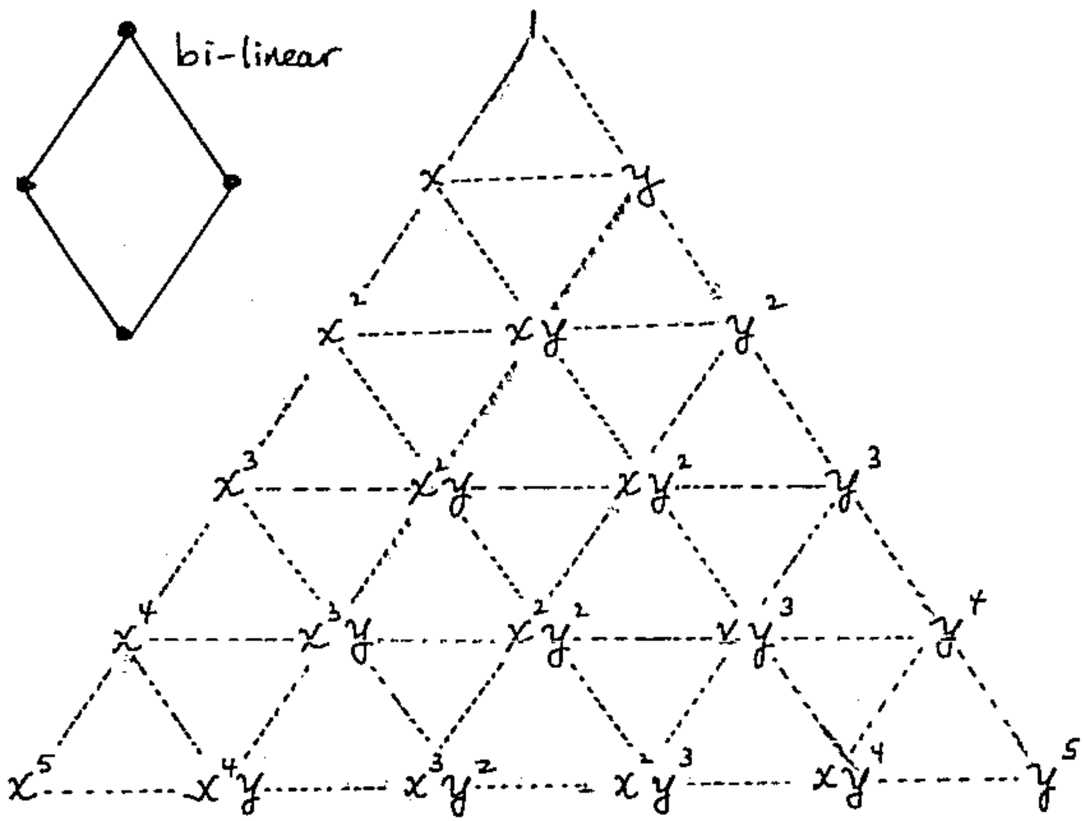
1

2

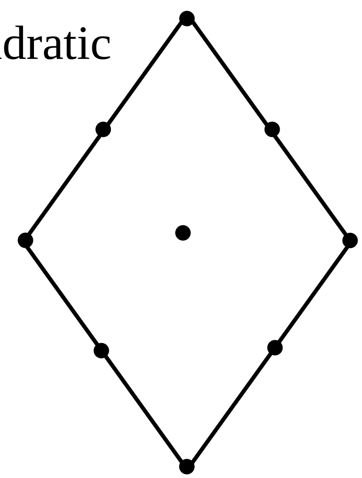
3

4

5



bi-quadratic



i. Multiply each row by x and y to get the terms in the next row

ii. vertices define nodes for a given quadrilateral

iii. generic interpolating polynomial for a given element involves all terms included within the defined quadrilateral.

Thus Pascal's triangle defines both the terms in the function in addition to the nodes involved.

2. Set constraints and solve.

If the nodes are defined as $(\xi_j, \eta_j) \quad j = 1, N$

we set the constraints:

$$\left.\begin{aligned} \phi_i(\xi_i, \eta_i) &= 1.0 \\ \phi_i(\xi_i, \eta_i) &= 0.0 \quad i \neq j \quad j = 1, N \end{aligned}\right\} i = 1, N$$

Thus we must solve for N N x N systems of equations.

Example: Bi-linear element

Step 1

Define the general form of the interpolating polynomial:

$$\phi_i = a_i + b_i\xi + c_i\eta + d_i\xi\eta$$

define nodes: $N = 4$

$$(\xi_j, \eta_j) = (-1,-1)$$
$$(-1,+1)$$
$$(+1,-1)$$
$$(+1,+1)$$

Step 2

Set constraints and set up $i = 1, N$ systems of equations:

$$\phi_i = (\xi_j, \eta_j) = \begin{cases} 1.0 & 1 = j \\ 0.0 & i \neq j \end{cases}$$

Now solve the systems of 4 equations.

- We obtain same $\underline{\phi}$ as we got multiplying the 1-D functions together. This procedure is however more general. It really is the same process as was used in 1-D cases, except that we use Pascal's triangle to help define the nodes and form of the interpolating functions.

  We note that the interior nodes defined for Lagrangian basis are not necessary for achieving inter-element functional continuity.

  Let's develop interpolating basis which do not use interior nodes.

Serendipity Basis Functions

- Bi-linear Lagrange Quadrilateral has no interior nodes and therefore we can not simplify this element
- Simplify Bi-quadratic Lagrange Quadrilateral by defining a set of 2-D bases that are quadratic along each side yet have no center node.

For a complete quadratic 2-D element, the form of the general inerpolating function is (using Pascal's triangle):

$$\phi_i = a_i + b_i\xi + c_i\eta + d_i\xi^2 + e_i\eta^2 + f_i\xi\eta + g_i\xi^2\eta + h_i\xi\eta^2 + p_i\xi^2\eta^2$$

Thus there are 9 coefficients and hence we require 9 constraints to derive an equation ϕ_i for each node i. This would lead us to the bi-quadratic element we established before.

Let's drop the last term in the expression for ϕ_i (i.e. set $p_i \equiv 0$). We are left with a polynomial with 8 coeffcients. Therefore we must also eliminate a node and we delete the interior node.

$$\phi_i = a_i + b_i\xi + c_i\eta + d_i\xi^2 + e_i\eta^2 + f_i\xi\eta + g_i\xi^2\eta + h_i\xi\eta^2$$

- Develop 8 functions by for each function:

  i. Setting $\phi_i = 1$ at the given node

  ii. Setting $\phi_i = 0$

  We can then derive the following interpolating functions:

  at corner nodes: $\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1)$

  at side nodes $\xi_i = 0$ $\frac{1}{2}(1-\xi^2)(1+\eta\eta_i)$

  at side nodes $\eta_i = 0$ $\frac{1}{2}(1+\xi\xi_i)(1-\eta^2)$

  There are no interior nodes:

  This element now has:

  - Quadratic variation along the sides (with full continuity of the function along inter-element boudnaries)
  - Cubic variation on the interior (vs. Quintic for the corresponding Bi-quadratic Lagrange element)

    However for the C-D equation, the use of the serendipity element results in a substantial loss in accuracy as compared to the bi-quadratic Lagrange basis. The increased economy which results due to 1 less node is not justified due to this severe degradation in accuracy.